

Manipulation of Social Decision Functions

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1. INTRODUCTION¹

A social decision function is a relational collective choice rule which assigns a complete, reflexive, and acyclic binary relation to each preference profile. In this paper we investigate the possibility that there exist satisfactory social decision functions which are free from strategic manipulation.

An individual can manipulate a social decision function if he can profitably alter society's choice by misrepresenting his preferences. Since social decision functions may select more than one alternative for given issues and preference profiles, whether or not a change in society's choice is profitable to an individual will depend not only on the individual's preferences over alternatives but also on his preferences over sets of alternatives. These in turn will depend on the way in which society makes its final selection out of sets of socially indifferent alternatives, as well as on the behavioral characteristics of individuals. It is assumed here that this selection is made by means of some random device among the alternatives of the choice set. An individual's preferences over sets of alternatives would then depend on the specific random device and on his attitude toward risk.

A social decision function is *uniformly manipulable* if there exist a preference profile and an issue for which it would be to some individual's advantage to misrepresent his preferences over alternatives *regardless of his particular attitude toward risk and of the particular random mechanism to be used in case of social ties*. Our main result (Theorem 2) proves that *all nonimposed social decision functions are either oligarchic or uniformly manipulable*. To extend

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¹ This section provides an informal description of the results. Our formal analysis begins in Section 2, where definitions are provided.

this result, we remark that even under oligarchic social decision functions some individuals may find it profitable to misrepresent their preferences, depending on their attitude toward risk. Theorem 3 proves that, in fact, all social decision functions will be vulnerable to strategy in a certain sense (although not always uniformly) unless they have a dictator.

Our main result provides a conclusive negative answer to the question whether one could find satisfactory relational collective choice rules that never yield empty choice sets. It thus appears that if a collective choice rule is to exist which is not manipulable either it will not be rationalizable by means of a binary criterion for social choice or else it will lead to empty choices for some issues and preference profiles.²

A number of preceding works have provided evidence that Arrow's conditions are crucially related to the possibility that a group decision rule be subject to manipulation [2, 5-7, 10, 11, 13, 14, 16, 20]. This relationship is further clarified along the proof of our main result. There we show that, on the set of strict profiles, binarity, monotonicity, and quasi-transitivity are necessary conditions for a social decision function not to be uniformly manipulable. These necessity results are important because they provide a new rationale for demanding from social decision functions these slightly modified versions of Arrow's conditions.

The result presented here is parallel to Gibbard's [5] and Satterthwaite's [16] conclusion that all voting schemes must be either dictatorial or manipulable (when their range contains more than two alternatives). It shows that the difficulties involved in the design of nonmanipulable mechanisms for social choice are not due to the specific kind of formalization—voting schemes—³ that they studied, but are also central to the standard formalization of social choice processes along the lines of Arrow's work.

2. NOTATION AND DEFINITIONS

Individuals. Let $J = \{1, 2, \dots, N\}$ be an initial segment of the positive integers. J is called the set of individuals, and N is the number of individuals.

Alternatives. Let X be a (finite or infinite) set. The elements of X are denoted by x, y, z, \dots and are called alternatives.

² Whether a nonrationalizable collective choice rule exists which is not manipulable and always leads to nonempty choices for nonempty finite issues is an open question. Results by Pattanaik [12], however, show that important classes of such rules are extremely manipulable. As for the manipulation of collective choice rules which may lead to empty choices, it is difficult to interpret the meaning of such cases where no alternative is chosen and, therefore, to give an appropriate definition about when such rules are manipulable.

³ Voting schemes assign a single alternative to each preference profile.

Issues. Let \mathcal{X} be the set of all nonempty subsets of X . Elements of \mathcal{X} are denoted by Y, Z, \dots and are called issues.

Choice criteria. Given a binary relation B on X , the strict relation \bar{B} associated with B is defined by

$$x\bar{B}y \leftrightarrow [xB y \wedge \sim (yBx)].$$

The indifference relation \tilde{B} associated with B is defined by

$$x\tilde{B}y \leftrightarrow (xB y \wedge yBx).$$

A binary relation B on X is

- (a) reflexive, iff $(\forall x \in X) xBx$,
- (b) complete, iff $(\forall x, y \in X)[x \neq y \rightarrow (xB y \vee yBx)]$,
- (c) asymmetric, iff $(\forall x, y \in X)[xB y \rightarrow \sim(yBx)]$,
- (d) transitive, iff $(\forall x, y, z \in X)[(xB y \wedge yBz) \rightarrow xBz]$,
- (e) quasi-transitive, iff $(\forall x, y, z \in X)[(x\bar{B}y \wedge y\bar{B}z) \rightarrow x\bar{B}z]$,
- (f) acyclic, iff

$$(\forall x_1, x_2, x_3, \dots, x_n \in X)[(x_1\bar{B}x_2 \wedge x_2\bar{B}x_3 \wedge \dots \wedge x_{n-1}\bar{B}x_n) \rightarrow x_1Bx_n].$$

Let \mathcal{B} be the set of complete, reflexive, and acyclic binary relations on X . Elements of \mathcal{B} are denoted by B, B' , and called choice criteria.

Choice sets. Given $B \in \mathcal{B}$ and $Y \in \mathcal{X}$, the choice set $C(B, Y)$ is defined by

$$C(B, Y) = \{y \in Y \mid (\forall z \in Y) yBz\}.$$

Since B is complete, reflexive, and acyclic, $C(B, Y)$ will be nonempty for all nonempty finite issues Y (see [17], Lemma 1*.1, p. 16).

Preferences. An ordering on X is a complete, reflexive, and transitive binary relation on X . Let \mathcal{R} be the set of orderings on X . Elements of \mathcal{R} are denoted by R, R' , and are called preferences (over alternatives). The strict preference relation P and the indifference relation I associated with a given R are defined in the customary way.

Preference profiles. Elements of the Cartesian product \mathcal{R}^N are denoted by \mathbf{R}, \mathbf{R}' , and are called preference profiles. Let $S \subset \mathcal{R}^N$ be the set of preference profiles where no individual is indifferent between any two alternatives. Elements of S are called strict preference profiles.

DEFINITION 1. A *social decision function* is a function from \mathcal{R}^N to \mathcal{B} . It is,

thus, a rule F that assigns a choice criterion to each preference profile.⁴ It is understood that when society's preference profile is $\mathbf{R} \in \mathcal{R}^N$, its decision on a given issue $Y \in \mathcal{X}$ will be given by the choice set $C(F(\mathbf{R}), Y)$.

3. A POSSIBILITY THEOREM FOR SOCIAL DECISION FUNCTIONS

We start this section by defining a number of standard conditions on social decision functions. We then state without proof a central theorem that will be of use later. We formulate these conditions for preference profiles in \mathcal{R}^N . On occasion, however, we find it appropriate to refer to the case where the same conditions hold for strict preference profiles only (i.e., for \mathbf{R} 's in S). When this is the case, we state explicitly that the property holds on S .

DEFINITION 2. A social decision function F is *nonimposed* iff

$$(\forall x, y \in X)[\exists \mathbf{R} \in \mathcal{R}^N: x\bar{B}y],$$

where \bar{B} is the strict relation associated with $B = F(\mathbf{R})$.

DEFINITION 3. A social decision function F is *binary* iff

$$(\forall x, y \in X)\{[(\forall i)(xR_i y \leftrightarrow xR'_i y)] \rightarrow (xB y \leftrightarrow xB' y)\},$$

where $B = F(\mathbf{R})$ and $B' = F(\mathbf{R}')$.

DEFINITION 4. A social decision function F is *monotonic* iff

$$(\forall x \in X)\{(\forall i \in J)[(\forall a, b \neq x)(aR_i b \leftrightarrow aR'_i b) \wedge (\forall a \in X)[(xP_i a \rightarrow xP'_i a) \wedge (xI_i a \rightarrow xR_i a)]] \rightarrow [(x\bar{B} y \rightarrow x\bar{B}' y) \wedge (x\tilde{B} y \rightarrow xB' y)],$$

where $B = F(\mathbf{R})$, $B' = F(\mathbf{R}')$.

DEFINITION 5. A social decision function F is *dictatorial* iff $\exists i \in J$ such that $(\forall x, y \in X)(xP_i y \rightarrow x\bar{B} y)$, where \bar{B} is the strict relation associated with $B = F(\mathbf{R})$. Individual i is then called the dictator.

DEFINITION 6. A social decision function F is *oligarchic* iff $\exists O \subset J$ such that

- (1) $(\forall \mathbf{R} \in \mathcal{R}^N)(\forall x, y \in X)[(i \in O \wedge xP_i y) \rightarrow xB y]$, where $B = F(\mathbf{R})$, and
- (2) $(\forall x, y \in X)[(\forall i \in O)(xP_i y) \rightarrow x\bar{B} y]$, where \bar{B} is the strict relation associated with $B = F(\mathbf{R})$.

⁴ Notice that built into our definition is the hypothesis that F operates over the universal domain.

In words, F is oligarchic iff there exists a group of individuals O such that, if any one of them prefers any alternative x to any y , society must regard x to be at least as good as y , and if all members of the group strictly prefer x to y , then society must strictly prefer x to y . O is then called an *oligarchy*, and its members are called *vetoers*.

Notice that an oligarchy, if it exists, must be unique. If it consists of a single individual, this individual is a dictator.

DEFINITION 7. A social decision function F is *quasi-transitive* iff $F(\mathbf{R})$ is a quasi-transitive binary relation for all $\mathbf{R} \in \mathcal{R}^N$.

DEFINITION 8. A social decision function F satisfies the *weak Pareto criterion* iff, for all $x, y \in X$, $[(\forall i \in J) xP_i y] \rightarrow x\bar{B}y$, where \bar{B} is the strict relation associated with $B = F(\mathbf{R})$.

We now state without proof a theorem that will be used later on. We formulate it for the case where all properties involved hold on S . It could also be stated for the case where these properties hold on \mathcal{R}^N .⁵

THEOREM 1. Let $|X| > 2$. If a social decision function F is nonimposed, binary, monotonic, and quasi-transitive on S , it is oligarchic on S .

4. EXTENDED PREFERENCES, CONSISTENT INDIVIDUALS, AND UNIFORMLY MANIPULABLE SOCIAL DECISION FUNCTIONS

We have assumed that individuals have preferences defined on alternatives, formalized by orderings on X . We now introduce the notion that they might regard some sets of alternatives as more desirable than others. We define individuals to be consistent when their strict preferences over sets of alternatives satisfy a certain minimal condition.

Let \mathcal{P}^* be the set of asymmetric and transitive binary relations on \mathcal{X} . Elements of \mathcal{P}^* are denoted by P^* and are called preferences over \mathcal{X} (or preferences over sets).⁶ Elements of the Cartesian product $\mathcal{R} \times \mathcal{P}^*$ are denoted by (R, P^*) and are called extended preferences. Individuals will be assumed to be endowed with extended preferences.

⁵ This result is due to Gibbard. Other versions of it were independently arrived at by Schwartz, Mas-Colell and Sonnenschein, and Guha (see [18, p. 14]).

⁶ Notice that $P^* \in \mathcal{P}^*$ need not be complete.

DEFINITION 9.⁷ We say that $P^* \in \mathcal{P}^*$ is consistent with $R \in \mathcal{R}$ iff, for all $x, y \in X$,

$$xPy \rightarrow (\langle x \rangle P^* \langle x, y \rangle \wedge \langle x, y \rangle P^* \langle y \rangle).$$

An individual i is consistent iff his extended preferences (R_i, P_i^*) are such that P_i^* is consistent with R_i .

It is often assumed that in cases where the social choice set consists of several alternatives, one of them is selected at random to become society's final choice. Under this interpretation, *all individuals endowed with any of the usual criteria for behavior under risk would always be consistent, regardless of the specific random mechanism to be used in the case of social ties.* This would be the case, for example, with all individuals whose preferences over sets of alternatives can be rationalized by a Von Neumann–Morgenstern utility function as well as those who follow the maximin rules described by Pattanaik in [11].

We are now ready to define uniformly manipulable social decision functions. The purpose of this definition is to concentrate attention on those situations where social decision functions are subject to unambiguous threats of being manipulated.

DEFINITION 10. A social decision function F is *uniformly manipulable* iff $\exists i \in J, Y \in \mathcal{X}$, and $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$ such that $(\forall j \neq i) R_j = R'_j$, and

$$C(F(\mathbf{R}'), Y) P_i^* C(F(\mathbf{R}), Y) \quad \text{for all } P_i^* \text{ consistent with } R_i.$$

To explain the meaning of this definition, let us first concentrate on a specific P_i^* consistent with R_i . Suppose that \mathbf{R} was indeed the preference profile corresponding to the true preferences of individuals. Individual i , endowed with extended preferences (R_i, P_i^*) could change the preference profile from \mathbf{R} to \mathbf{R}' by misrepresenting his preferences to be R'_i . Suppose i had reason to believe that Y was to be the set of feasible alternatives. Then he could carry out the misrepresentation we just described and cause society's choice to be a better one for him than if he was to reveal his true preferences R_i . That fact in itself can already be thought of as a threat to F 's immunity to strategy. Still, our definition of uniform manipulability involves the further requirement that this threat should be advantageous to all consistent individuals (R_i, P_i^*) . By imposing this requirement we restrict our attention only to

⁷ Our consistency requirement is strongly reminiscent of Luce and Raiffa's [8, p. 28] Condition 6 (monotonicity) for the existence of a Von Neumann–Morgenstern utility function conforming to an individual's preferences. Professor Fishburn called my attention to his paper on "Even Chance Lotteries in Social Choice Theory" [3] where a number of conditions on individual preferences over sets of alternatives are postulated. Fishburn's conditions are stronger, but are in the same spirit, than our consistency requirement.

those situations where an unambiguous threat arises, i.e., one that it would be in the interest of *any* consistent individual to carry out, regardless of his further attitude toward risk.⁸ Even if a social decision function is not uniformly manipulable, some consistent individuals may still find it profitable to misrepresent their preferences. When it is uniformly manipulable, then it is indeed extremely vulnerable to strategy.

5. THE POSSIBILITY OF SOCIAL DECISION FUNCTIONS WHICH ARE NOT UNIFORMLY MANIPULABLE

THEOREM 2. *Let $|X| > 2$. A nonimposed social decision function is either uniformly manipulable or oligarchic.*

Proof. The proof is organized in five steps. We assume that F is a non-imposed social choice function and that it is not uniformly manipulable. Step 1 then establishes that F must be binary on S , Step 2 that it must be monotonic and Step 3 that it must be quasi-transitive. Therefore, by Theorem 1, F is oligarchic on S . Step 4 establishes that if F is oligarchic on S , it must be oligarchic on \mathcal{P}^N . Step 5 summarizes the argument.

Step 1. (If F is not uniformly manipulable, then it is binary on S .) Assume, a contrario, that F is not binary on S . Then, $\exists \mathbf{R}, \mathbf{R}' \in S, x, y \in X$ such that

$$(\forall i)(xR_i y \leftrightarrow xR'_i y),$$

and yet

$$C(F(\mathbf{R}), \langle x, y \rangle) \neq C(F(\mathbf{R}'), \langle x, y \rangle).$$

Therefore, there must exist some j ($1 \leq j \leq N$) such that

$$\mathbf{R}^j = (R'_1, \dots, R'_{j-1}, R_j, R_{j+1}, \dots, R_N),$$

$$\mathbf{R}^{j+1} = (R'_1, \dots, R'_{j-1}, R'_j, R_{j+1}, \dots, R_N),$$

and $C(F(\mathbf{R}^j), \langle x, y \rangle) \neq C(F(\mathbf{R}^{j+1}), \langle x, y \rangle)$. Suppose, without loss of generality, that $x \in C(F(\mathbf{R}^j), \langle x, y \rangle)$ and that $x \notin C(F(\mathbf{R}^{j+1}), \langle x, y \rangle) = \langle y \rangle$. If $y P_j x$, then $C(F(\mathbf{R}^{j+1}), \langle x, y \rangle) P_j^* C(F(\mathbf{R}^j), \langle x, y \rangle)$ for all P_j^* consistent with R_j . Thus, F would be uniformly manipulable, contrary to our assumption. If $x P_j y$, then, by assumption, $x P'_j y$. Therefore, $C(F(\mathbf{R}^j), \langle x, y \rangle) P_j^* C(F(\mathbf{R}^{j+1}), \langle x, y \rangle)$ for all P_j^* that are consistent with R'_j , and F would again be

⁸ Since consistent individuals may have incomplete preferences over sets of alternatives, for a social decision function to be uniformly manipulable there must exist a manipulation that individuals whose preferences are defined only over one- and two-element sets (in particular) can find profitable. This actually means that we restrict attention to the manipulations for which $C(F(\mathbf{R}), Y) \cup C(F(\mathbf{R}'), Y)$ is a two-element set.

uniformly manipulable. Since, by assumption, $\mathbf{R} \in S$, these two cases exhaust all possibilities, and this establishes Step 1.

Step 2. (If F is not uniformly manipulable, it is monotonic on S .) Suppose, a contrario, that F is not monotonic. Then, there exist $\mathbf{R}, \mathbf{R}' \in S, x, y \in X$ such that

$$(\forall i \in J)\{(\forall a, b \neq x)(aR_i b \leftrightarrow aR'_i b) \wedge (\forall a \in X)(xP_i a \rightarrow xP'_i a)\},$$

and yet either

$$x\bar{B}y \text{ and } yB'x \quad \text{or} \quad x\bar{B}'y \text{ and } y\bar{B}x,$$

where $F(\mathbf{R}) = B$ and $F(\mathbf{R}') = B'$. There must then exist $j (1 \leq j \leq N)$ such that

$$\begin{aligned} \mathbf{R}^j &= (R'_1, \dots, R'_{j-1}, R_j, R_{j+1}, \dots, R_N), \\ \mathbf{R}^{j+1} &= (R'_1, \dots, R'_{j-1}, R'_j, R_{j+1}, \dots, R_N), \end{aligned}$$

and yet, either

- (1) $x\bar{B}_j y$ and $yB_{j+1}x$, or
- (2) $x\bar{B}'_j y$ and $y\bar{B}_{j+1}x$,

where $F(\mathbf{R}^j) = B_j$ and $F(\mathbf{R}^{j+1}) = B_{j+1}$. If $yP_j'x$, then it must be that yP_jx . For all P_j^* consistent with R_j we have $C(B_{j+1}, \langle x, y \rangle) P_j^* C(B_j, \langle x, y \rangle)$, whether (1) or (2) hold. Therefore, F is uniformly manipulable, contrary to our assumption. If $xP_j'y$, then for all P_j^* consistent with R'_j we have $C(B_j, \langle x, y \rangle) P_j^* C(B_{j+1}, \langle x, y \rangle)$, whether (1) or (2) hold, and F is again uniformly manipulable, contrary to our assumption. The above contradictions establish that F must be monotonic and complete the proof of Step 2.

Step 3. (If F is not uniformly manipulable, it is quasi-transitive on S .) Suppose F is not quasi-transitive on S . Then there exist $\mathbf{R} \in S$ and $x, y, z \in X$ such that $x\bar{B}y, y\bar{B}z$, and $z\bar{B}x$, where \bar{B} and \bar{B}' , respectively, are the strict preference and indifference relations associated with $B = F(\mathbf{R})$. Hence, $C(B\langle x, y, z \rangle) = \langle x \rangle$. Now construct $\mathbf{R}' \in S$ as follows: For all $i \in J, xP'_i z$ iff $xP_i z$, and for all $a \in X$ distinct from x and $z, xP'_i a$ and $zP'_i a$. Since F is nonimposed, binary, and monotonic (from earlier steps), F satisfies the weak Pareto criterion (see [1, Theorem 1, p. 97]), and hence for all $a \in X$ distinct from x and $z, x\bar{B}'a$ and $z\bar{B}'a$, where \bar{B}' is the strict preference relation corresponding to $B' = F(\mathbf{R}')$. Also, by binarity of $F, x\bar{B}'z$. Thus, $C(F(\mathbf{R}'), \langle x, y, z \rangle) = \langle x, z \rangle$. Therefore, and since F is monotonic, there must exist $j (1 \leq j \leq N)$ such that

$$\begin{aligned} C(F(\mathbf{R}'), \langle x, y, z \rangle) \\ = C(F(R'_1, \dots, R'_{j-1}, R_j, R_{j+1}, \dots, R_N), \langle x, y, z \rangle) = \langle x \rangle, \end{aligned}$$

and

$$C(F(\mathbf{R}^{j+1}), \langle x, y, z \rangle) \\ = C(F(R'_1, \dots, R'_{j-1}, R'_j, R_{j+1}, \dots, R_N), \langle x, y, z \rangle) = \langle x, z \rangle.$$

If xP_jz , then xP'_jz by construction and j can manipulate at \mathbf{R}^{j+1} by switching from R'_j to R_j . In case zP_jx , then h can manipulate at \mathbf{R}^j by switching from R_j to R'_j . It is clear that in either case F would be uniformly manipulable, a contradiction. Therefore, F must be quasi-transitive on S .

Step 4. (If O is an oligarchy for F on S , it is an oligarchy for F on \mathcal{R}^N .) We must show that

- (a) all vetoers on S are vetoers (on \mathcal{R}^N), and
 (b) for all alternatives x and y and all preference profiles, whenever all individuals in O strictly prefer x to y , society ranks x as strictly preferred to y .

(a) Assume h is a vetoer on S , but not on \mathcal{R}^N , i.e., $\exists \mathbf{R} \in \mathcal{R}^N$ for which xP_hy , $B = F(\mathbf{R})$, and yBx . In that case, $C(F(\mathbf{R}), \langle x, y \rangle) = \langle y \rangle$. Let T be a strong ordering on X for which yTx , and let \mathbf{R}' be defined by

$$(\forall i \in J)(\forall z, w \in X)\{zP'_i w \leftrightarrow [zP_i w \vee (zI_i w \wedge zTw)]\}.$$

Informally, \mathbf{R}' is obtained from \mathbf{R} by breaking the ties that appear in \mathbf{R} according to the ordering T . By construction, $\mathbf{R}' \in S$ and $xP_h y$. Let $B' = F(\mathbf{R}')$. Since h is a vetoer on S , we have that $xB'y$. Hence, $x \in C(F(\mathbf{R}'), \langle x, y \rangle)$, while $x \notin C(F(\mathbf{R}), \langle x, y \rangle) = \langle y \rangle$. Thus, there must exist a j ($1 \leq j \leq N$) such that

$$x \notin C(F(\mathbf{R}^j), \langle x, y \rangle) = C(F(R'_1, \dots, R'_{j-1}, R_j, R_{j+1}, \dots, R_N), \langle x, y \rangle)$$

and

$$x \in C(F(\mathbf{R}^{j+1}), \langle x, y \rangle) = C(F(R'_1, \dots, R'_{j-1}, R'_j, R_{j+1}, \dots, R_N), \langle x, y \rangle).$$

If yR_jx , since yTx , we have that yP'_jx . Thus, $C(F(\mathbf{R}^j), \langle x, y \rangle) = P_j^*C(F(\mathbf{R}^{j+1}), \langle x, y \rangle)$ for all P_j^* consistent with R'_j , and F is uniformly manipulable. If xP_jy , then $C(F(\mathbf{R}^{j+1}), \langle x, y \rangle) = P_j^*C(F(\mathbf{R}^j), \langle x, y \rangle)$ for all P_j^* consistent with R_j , and F is uniformly manipulable. In both cases we get a contradiction. Thus, h must be a vetoer on \mathcal{R}^N if it is a vetoer on S .

(b) Assume $\exists x, y \in X$, $\mathbf{R} \in \mathcal{R}^N$ such that $(\forall i \in O) xP_i y$, and yet yBx , where $B = F(\mathbf{R})$. Let T be an arbitrary strong ordering for which yTx , and define $\mathbf{R}' \in S$ so that $(\forall i \in J)(\forall z, w \in X)\{zP'_i w \leftrightarrow [zP_i w \vee (zI_i w \wedge zTw)]\}$. Informally, \mathbf{R}' is obtained from \mathbf{R} by breaking the ties that appear in \mathbf{R}

according to the ordering T . By construction, $\mathbf{R}' \in S$ and $xP_i y$ for all $i \in O$. Let $B' = F(\mathbf{R}')$. Since O is an oligarchy on S , we have that $x\bar{B}'y$. Hence, $\langle x \rangle = C(F(\mathbf{R}'), \langle x, y \rangle)$, while $y \in C(F(\mathbf{R}), \langle x, y \rangle)$. Thus, there must exist $j (1 \leq j \leq N)$ such that

$$y \in C(F(\mathbf{R}^j), \langle x, y \rangle) = C(F(R'_1, \dots, R'_{j-1}, R_j, R_{j+1}, \dots, R_N), \langle x, y \rangle),$$

$$y \notin C(F(\mathbf{R}^{j+1}), \langle x, y \rangle) = C(F(R'_1, \dots, R'_{j-1}, R'_j, R_{j+1}, \dots, R_N), \langle x, y \rangle).$$

If $xP_j y$, then $C(F(\mathbf{R}^{j+1}), \langle x, y \rangle) P_j^* C(F(\mathbf{R}^j), \langle x, y \rangle)$ for all P_j^* consistent with R_j , and F is uniformly manipulable. If $yP_j x$, then $yP_j' x$ by construction and $C(F(\mathbf{R}^j), \langle x, y \rangle) P_j^* C(F(\mathbf{R}^{j+1}), \langle x, y \rangle)$ for all P_j^* consistent with R'_j . Thus, F is uniformly manipulable in this case too. In both cases we get a contradiction. Therefore, O must be an oligarchy (on \mathcal{R}^N) if it is an oligarchy on S .

Step 5. (If F is not uniformly manipulable, it is oligarchic.) Since F is not uniformly manipulable, it is binary, monotonic, and quasi-transitive on S , by Steps 1, 2, and 3. Thus, according to Theorem 1, F is oligarchic on S . But then, according to Step 4, F is oligarchic. This completes the proof of the Theorem.

In closing this section, notice that in Step 1 of the proof of Theorem 2 we have shown that binary on S (the set of strict preference profiles) is necessary for a social decision function not to be uniformly manipulable. The following example shows that binary on the set of all preference profiles would not be necessary.

EXAMPLE 1. Let F be defined as follows. For any $\mathbf{R} \in \mathcal{R}^N$, $F(\mathbf{R}) = B_{\mathbf{R}}$, where $B_{\mathbf{R}} = R_N$ for all $\mathbf{R} \in \mathcal{R}^N$ such that $C(R_N, x) \neq \langle x, y \rangle$, and $[x\bar{B}_{\mathbf{R}}y \wedge (\forall \langle z, w \rangle \neq \langle x, y \rangle)(zB_{\mathbf{R}}w \leftrightarrow zR_Nw)]$ for all $\mathbf{R} \in \mathcal{R}^N$ such that $C(R_N, x) = \langle x, y \rangle$.

In this example, N is a dictator for F . Also, whenever N is indifferent among some alternatives, society is also indifferent among these, except for the case where N is indifferent between x and y and prefers these to all other alternatives. In this case x is strictly preferred to y by society. The reader may check that F is neither binary nor uniformly manipulable on \mathcal{R}^N .

6. DISCERNING INDIVIDUALS AND THE MANIPULATION OF OLIGARCHIC SOCIAL DECISION FUNCTIONS

In this section we define the concept of social decision functions which are manipulable by discerning individuals. In Theorem 3 we prove that all social decision functions which are not manipulable by discerning individuals

must be dictatorial. This theorem reaches a stronger conclusion than Theorem 2 by using a more demanding strategy-proofness requirement. To motivate our analysis, we first provide an example of an oligarchic social decision function under which individuals endowed with certain attitudes toward risk would find it profitable to misrepresent their preferences. Theorem 3 proves that this will be the case for all social decision functions unless they have a dictator.

EXAMPLE 2. (The weak Pareto-Extension Rule). Let F be defined so that $xBy \leftrightarrow [(\forall i) xI_i y \vee (\exists j \in xP_i y)]$. Let a, b , and c be any three alternatives in X . Consider the preference profile \mathbf{R} , where all individuals except N strictly prefer a to b and b to c , and N strictly prefers c to b and b to a . Then for preference profile \mathbf{R} , F would rank a, b and c as socially indifferent. Suppose that N would declare to prefer c to a and a to b . Let \mathbf{R}' be the resulting preference profile. Given \mathbf{R}' , F would rank a as socially indifferent to c , b as socially indifferent to c , and a as strictly preferred to b . Thus,

$$C(F(\mathbf{R}'), \langle a, b, c \rangle) = \langle a, c \rangle, \quad \text{while} \quad C(F(\mathbf{R}), \langle a, b, c \rangle) = \langle a, b, c \rangle.$$

Suppose, to be specific, that the random mechanism to be used assigns the same probability to all elements of the choice set. For the profile \mathbf{R} , individual N could determine the social choice to be either $\langle a, b, c \rangle$ or $\langle a, c \rangle$, by either declaring his actual preferences or misrepresenting them. If N was a maximizer of the type considered by Pattanaik or an expected utility maximizer with $u(a) + u(c) > 2u(b)$, it would be profitable for him to misrepresent his preferences at \mathbf{R} under the given social decision function.

The above example motivates the following definitions.

DEFINITION 11. A *dominance criterion* is a function d from \mathcal{R} to \mathcal{P}^* .

DEFINITION 12. A dominance criterion is *consistent* iff for all $R \in \mathcal{R}$, $d(R) = P^*$ is consistent with R . A dominance criterion is *discerning* iff it is consistent and, for each $R \in \mathcal{R}$ and any $x, y, z \in X$,

$$xPyPz \rightarrow (\langle x, y, z \rangle P^* \langle x, z \rangle \vee \langle x, z \rangle P^* \langle x, y, z \rangle), \quad \text{where } P^* = d(R).$$

Let \mathcal{D} be the set of discerning dominance criteria.

DEFINITION 13. A social decision function is *manipulable by discerning individuals* iff, for each $d \in \mathcal{D}$, there exist $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$, $i \in J$, and $Y \in \mathcal{X}$ such that

$$(\forall j \neq i) R_j = R'_j \quad \text{and} \quad C(F(\mathbf{R}'), Y) P_i^* C(F(\mathbf{R}), Y), \quad \text{where } P_i^* = d(R_i).$$

THEOREM 3. Let $|X| > 2$. If a nonimposed social decision function F is not manipulable by discerning individuals, then it is dictatorial.

Proof. Since F is not manipulable by discerning individuals, it is not uniformly manipulable. Thus, it is oligarchic. If the oligarchy consists of a single individual, such individual is a dictator. Therefore, it suffices to prove that there cannot be more than one vetoer. To show this, we prove that if F has two vetoers it is manipulable by discerning individuals, a contradiction.

Suppose that m and n are vetoers for F . Let d be any discerning dominance criterion. Let R be a strict preference relation such that $xPyPz$, and $P^* = d(R)$. Since d is discerning, either (1) $\langle x, y, z \rangle P^* \langle x, z \rangle$ or (2) $\langle x, z \rangle P^* \langle x, y, z \rangle$. If (1), let $\mathbf{R} \in \mathcal{R}^N$ be such that $R_n = R$ and $(\forall i \neq n) zP_i xP_i y$. Since both m and n are vetoers and F is oligarchic, $C(F(\mathbf{R}), \langle x, y, z \rangle) = \langle x, z \rangle$. Now, consider $\mathbf{R}' \in \mathcal{R}^N$, where $(\forall i \neq n) R_i = R_i'$ and $yP_n' xP_n' z$. Since both n and m are vetoers, $C(F(\mathbf{R}'), \langle x, y, z \rangle) = \langle x, y, z \rangle$. Therefore, since by construction $P_n^* = d(R_n) = P^*$, $C(F(\mathbf{R}'), \langle x, y, z \rangle) P_n^* C(F(\mathbf{R}), \langle x, y, z \rangle)$. If (2) holds, let $\mathbf{R} \in \mathcal{R}^N$ be such that $R_n = R$ and $(\forall i \neq n) zP_i yP_i x$. Since both n and m are vetoers, $C(F(\mathbf{R}), \langle x, y, z \rangle) = \langle x, y, z \rangle$. Consider $\mathbf{R}' \in \mathcal{R}^N$ such that $(\forall i \neq n) R_i = R_i'$ and $xP_n' zP_n' y$. Given our assumptions $C(F(\mathbf{R}'), \langle x, y, z \rangle) P_n^* C(F(\mathbf{R}), \langle x, y, z \rangle)$, because $C(F(\mathbf{R}'), \langle x, y, z \rangle) = \langle x, z \rangle$ and $P_n^* = d(R_n) = P^*$.

Cases (1) and (2) cover all possibilities that may arise for any discerning dominance criterion. In both cases we have described profiles \mathbf{R} and preference misrepresentations R_n' that would be profitable for individual n at \mathbf{R} . This contradicts the assumption that F is not manipulable by discerning individuals and completes the proof of Theorem 3.

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